| Subject | Class | Paper | Reading materials on | Resource Person | |
|---|---|---|---|---|---|
| Mathematics | D-I (H) | I | Theory of Equation | Dr. S. Ahmad | Associate Professor |

Problem Prove that the equation $\frac{A^2}{x-a}+\frac{B^2}{x-b}+\frac{C^2}{x-c}+\cdots+\frac{K^2}{x-k}=x+l$

Where $A, B, C, \ldots, a, b, c \ldots$ and $l$ are all real numbers has all its root real.

Soln. Let $\alpha+i\beta$ be a root of the given equation. Then $\alpha-i\beta$ will be also a root.

Therefore given equation become

$$\frac{A^2}{(\alpha+i\beta)-a}+\frac{B^2}{(\alpha+i\beta)-b}+\frac{C^2}{(\alpha+i\beta)-c}+\cdots+\frac{K^2}{(\alpha+i\beta)-k}=\alpha+i\beta+l$$

and
$$\frac{A^2}{(\alpha-i\beta)-a}+\frac{B^2}{(\alpha-i\beta)-b}+\frac{C^2}{(\alpha-i\beta)-c}+\cdots+\frac{K^2}{(\alpha-i\beta)-k}=\alpha-i\beta+l$$

Substracting above two relations we get

$$-2i\beta\left[\frac{A^2}{(\alpha-a)^2+\beta^2}+\frac{B^2}{(\alpha-b)^2+\beta^2}+\frac{C^2}{(\alpha-c)^2+\beta^2}+\cdots+\frac{K^2}{(\alpha-k)^2+\beta^2}\right]=2i\beta$$

$$\Rightarrow -2i\beta\left[\frac{A^2}{(\alpha-a)^2+\beta^2}+\frac{B^2}{(\alpha-b)^2+\beta^2}+\frac{C^2}{(\alpha-c)^2+\beta^2}+\cdots+\frac{K^2}{(\alpha-k)^2+\beta^2}+1\right]=0$$

The expression within the brackets is the sum of positive quantities and as such cannot be zero & hence $2i\beta=0$ $\Rightarrow \beta=0$. Therefore the given equation cannot have imaginary roots. Hence all its roots are real.

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Problem: Find the Condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ may be in (i) A.P. (ii) G.P. (iii) H.P.

Solution.

(i) Let the roots of the given equation be $a-d,\ a,\ a+d$ being in A.P.

$\therefore$ Sum of roots $= (a-d) + a + (a+d) = p$

$\Rightarrow 3a = p \quad \therefore a = \frac{p}{3}$.

Since $a$ is a root of the equation $x^3 - px^2 + qx - r = 0$

$$\therefore \left(\frac{p}{3}\right)^3 - p\cdot\left(\frac{p}{3}\right)^2 + q\cdot\frac{p}{3} - r = 0$$

$$\Rightarrow \frac{p^3}{27} - \frac{p^3}{9} + \frac{pq}{3} - r = 0$$

$$\Rightarrow p^3 - 3p^3 + 9pq - 27r = 0$$

$$\Rightarrow -2p^3 + 9pq - 27r = 0$$

$$\Rightarrow \boxed{2p^3 - 9pq + 27 = 0}$$

It is required Condition.

(ii) Let the roots of the equation be $\frac{a}{\rho}, a, a\rho$ being in G.P.

Product of the roots $= \Sigma\alpha\beta\gamma = \alpha\beta\gamma = \frac{a}{\rho}\cdot a\cdot a\rho = r$

$\Rightarrow a^3 = r$

But $a$ is root of the given equation

$\therefore a^3 - pa^2 + qa - r = 0$

$\Rightarrow \cancel{r} - pa^2 + qa - \cancel{r} = 0$

$\Rightarrow qa = pa^2$

$\Rightarrow pa = q \Rightarrow p^3a^3 = q^3 \Rightarrow p^3 r = q^3$

$\Rightarrow \boxed{p^3 r - q^3 = 0}$ It is required Condition

(iii) Let the roots be $\alpha, \beta, \gamma$ Which are in H.P and hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are in A.P $\therefore \frac{1}{\alpha} + \frac{1}{\gamma} = \frac{2}{\beta} \Rightarrow \beta\gamma + \alpha\beta = 2\gamma\alpha$

$\Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = 3\gamma\alpha$

$\Rightarrow q = 3\gamma\alpha \Rightarrow \gamma\alpha = \frac{q}{3}$.

Also $\alpha\beta\gamma = r \Rightarrow \gamma\alpha \cdot \beta = r \Rightarrow \frac{q}{3} \cdot \beta = r$

$\therefore \beta = \frac{3r}{q}$ & $\beta$ is a root of the given eqn.

$\beta^3 - p\beta^2 + q\beta - r = 0$

$\left(\frac{3r}{q}\right)^3 - p \cdot \left(\frac{3r}{q}\right)^2 + \cancel{q} \cdot \frac{3r}{\cancel{q}} - r = 0$

$\Rightarrow \frac{27r^3}{q^3} - \frac{9pr^2}{q^2} + 3r - r = 0$

$\Rightarrow 27r^3 - 9pr^2q + 2rq^3 = 0$

$\Rightarrow r(27r^2 - 9prq + 2q^3) = 0$, But $r \neq 0$

$\Rightarrow \boxed{27r^2 - 9pqr + 2q^3 = 0}$ It is required Condition.